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COMMENT

# Some recent results on symmetries of Lagrangian systems re-examined

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**Abstract.** Hojman's results on alternative Lagrangians and first integrals related to symmetries of Lagrangian systems are shown to be covered by previous treatments in a more geometrical setting. It is argued that the use of Lagrangians, linear in the accelerations, for second-order systems does not widen the spectrum of results.

## 1. Introduction

Many recent papers deal with certain aspects of the study of symmetries of Lagrangian systems. The construction of first integrals out of known symmetries is of course best known in the context of the celebrated Noether theorem and it is only fairly recently that one has learned to cope with other than Noether symmetries. Perhaps the first contribution in that direction was made by Lutzky (1979), who observed that point symmetries which are not of the Noether type produce a new Lagrangian for the system and that the existence of two Lagrangians leads to a matrix whose trace is a constant of the motion. Prince (1983), searching for a generalisation, identified a compatibility condition under which a velocity-dependent symmetry equally produces an alternative Lagrangian. Sarlet (1983) showed how, conversely, two different Lagrangians for the same system define a pair of symmetries, which in turn give rise to a pair of related first integrals. Further results regarding equivalent Lagrangians and matrices whose trace is conserved were derived by Hojman and Harleston (1981) and later confirmed by many other authors. In a more geometrical setting, Henneaux (1981) and Crampin (1983a) showed how non-Noether constants of this type can all be related to a type (1, 1) tensor field whose Lie derivative with respect to the given second-order vector field vanishes. In passing, Crampin remarked that a similar construction can be carried out starting from an arbitrary dynamical symmetry of the Lagrangian system, i.e. irrespective of the possible existence of an alternative Lagrangian.

Symmetries of Lagrangian systems and the construction of related first integrals is again the central theme in a recent paper by Hojman (1984). His approach is entirely classical and analytical and, at first sight, would appear to rely heavily on the use of Lagrangians which are linear in the accelerations. The principal aim thereby is to achieve a result, believed to be new, concerning general dynamical symmetries of the equations of motion. Because of the difference in approach, it is a non-trivial exercise to compare Hojman's results with those quoted above. We therefore think it useful to point out explicitly here how each of the statements in Hojman (1984) is actually covered by a result in one or more of the papers cited above. This will mean that the use of Lagrangians, linear in the accelerations, is quite redundant so far as ordinary second-order differential equations are concerned.

## 2. The redundancy of acceleration dependent Lagrangians

Throughout this paper we will be dealing with a system of second-order ordinary differential equations

$$\ddot{q}^i - \Lambda^i(t, q, \dot{q}) = 0 \qquad i = 1, \ldots, n \tag{1}$$

which is associated with the vector field

$$\Gamma = \partial/\partial t + \dot{q}^i\, \partial/\partial q^i + \Lambda^i\, \partial/\partial \dot{q}^i \tag{2}$$

on the manifold $\mathbb{R} \times TM$ with local coordinates $(t, q^i, \dot{q}^i)$. The evolution space $\mathbb{R} \times TM$ may be identified with the first jet bundle $J^1(\mathbb{R}, M)$ (see e.g. Crampin *et al* 1984). One might consider passing to higher-order jet bundles and see whether (1) can be derived from a Lagrangian $\tilde{L}(t, q, \dot{q}, \ddot{q})$. Generically, such a Lagrangian will give rise to differential equations of order 4. When $\tilde{L}$ is linear in $\ddot{q}$, say

$$\tilde{L} = \beta_i(t, q, \dot{q})\ddot{q}^i + F(t, q, \dot{q}) \tag{3}$$

the associated Euler-Lagrange equations will be of order 3 and they can only be of second order if we have $\beta_i = \partial f/\partial \dot{q}^i$ for some function $f$. It then follows, however, that

$$\tilde{L} \equiv L + \mathrm{d}f/\mathrm{d}t \tag{4}$$

where $L$ is defined by

$$L(t, q, \dot{q}) = F - \partial f/\partial t - \dot{q}^k\, \partial f/\partial q^k \tag{5}$$

and will be a conventional Lagrangian for the same system (1). When one insists that $\tilde{L}$ be a linear combination of its own equations of motion, in other words that the function $F$ in (3) be of the form $F = -\Lambda^i\, \partial f/\partial \dot{q}^i$, the expression (5) for the related conventional Lagrangian simply becomes

$$L = -\Gamma(f). \tag{6}$$

These simple arguments sufficiently indicate that nothing new is to be expected from using acceleration dependent Lagrangians in the study of (1), since every result about $\tilde{L}$ must have an immediate translation to a corresponding result for $L$.

For completeness, let us formulate the following alternative description of the situation. In a more modern framework, the Euler-Lagrange equations correspond to a characteristic vector field of the contact form $\mathrm{d}\theta$, where $\theta$ is the Poincaré-Cartan 1-form. In general, when it concerns a Lagrangian $\tilde{L}$ dependent on accelerations, the local expression of the appropriate Poincaré-Cartan form $\tilde{\theta}_{\tilde{L}}$ is given by (see e.g. Krupka 1983, Klapka 1983),

$$\tilde{\theta}_{\tilde{L}} = \tilde{L}\, \mathrm{d}t + \left(\frac{\partial \tilde{L}}{\partial \dot{q}^j} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tilde{L}}{\partial \ddot{q}^j}\right)\right)(\mathrm{d}q^j - \dot{q}^j\, \mathrm{d}t) + \frac{\partial \tilde{L}}{\partial \ddot{q}^j}(\mathrm{d}\dot{q}^j - \ddot{q}^j\, \mathrm{d}t). \tag{7}$$

In the case when $\tilde{L}$ is of the form (3), it is straightforward to verify that $\tilde{\theta}_{\tilde{L}}$ actually reduces to $\tilde{\theta}_{\tilde{L}} = \theta_L + \mathrm{d}f$, where $\theta_L$ is the conventional Cartan form:

$$\theta_L = L\,\mathrm{d}t + \frac{\partial L}{\partial \dot{q}^i}(\mathrm{d}q^i - \dot{q}^i\,\mathrm{d}t) \tag{8}$$

$L$ being defined by (5). So the difference between $\tilde{\theta}_{\tilde{L}}$ and $\theta_L$ turns out to be an exact 1-form, which does not play any significant role.

Let us now turn our attention to symmetries of the given system (1). A vector field $Y$, which has the local coordinate expression

$$Y = \mu^i(t, q, \dot{q})\,\partial/\partial q^i + \nu^i(t, q, \dot{q})\,\partial/\partial \dot{q}^i \tag{9}$$

represents a general dynamical symmetry of (1) if $[Y, \Gamma] = 0$. One might incorporate a $\partial/\partial t$ term in $Y$ and require the Lie bracket with $\Gamma$ to produce a multiple of $\Gamma$. However, as has repeatedly been argued in previous publications (see e.g. Sarlet 1983), it is only a certain equivalence class of symmetries that matters and one can work with the representation (9) without loss of generality. For the sake of comparison, recall that the symmetry condition in coordinates means

$$\nu^i = \Gamma(\mu^i) \qquad \text{and} \qquad \Gamma(\nu^i) = Y(\Lambda^i) \tag{10}$$

and note that (10) is identical to the symmetry requirement (2.6) in Hojman (1984).

Suppose now that $L$ is a Lagrangian governing (1), so that

$$i_\Gamma\,\mathrm{d}\theta_L = 0. \tag{11}$$

One may wonder under what circumstances the Lie derivative of $\mathrm{d}\theta_L$ with respect to the symmetry $Y$ will produce a new Cartan 2-form (not necessarily of maximal rank)

$$\mathscr{L}_Y\,\mathrm{d}\theta_L = \mathrm{d}\theta_{L'} \tag{12}$$

for some function $L'$. A concise formulation of the conditions for a 2-form $\Omega$ to be a Cartan form has been given by Crampin *et al* (1984) (theorem 2). With $\Omega = \mathscr{L}_Y\,\mathrm{d}\theta_L$, it is obvious that $\mathrm{d}\Omega = 0$ and $i_\Gamma\Omega = 0$ in view of (11) and the fact that $Y$ and $\Gamma$ commute. The only requirement which then remains to be satisfied is $\mathscr{L}_Y\,\mathrm{d}\theta_L(\partial/\partial\dot{q}^i, \partial/\partial\dot{q}^j) = 0$. When expressed in coordinate form, it reads

$$\alpha_{ij}\,\partial\mu^j/\partial\dot{q}^k = \alpha_{kj}\,\partial\mu^j/\partial\dot{q}^i \tag{13}$$

with

$$\alpha_{ij} = \partial^2 L/\partial\dot{q}^i\,\partial\dot{q}^j \tag{14}$$

by which we recover the compatibility condition identified by Prince (1983). Locally, (13) is equivalent to the condition $\alpha_{ij}\mu^j = -\partial G/\partial\dot{q}^i$ for some function $G(t, q, \dot{q})$, and as pointed out, e.g. by Sarlet (1983) (lemma 2), the new function $L'$ of (12) is then determined by $L' = -\Gamma(G)$. From the observations (3)–(6) it follows that a corresponding acceleration dependent Lagrangian is given by

$$\tilde{L}' = -\alpha_{ij}\mu^j(\ddot{q}^i - \Lambda^i). \tag{15}$$

It is therefore not a surprise that Hojman, in studying the conditions under which (15) provides a new Lagrangian (see his equation (4.2)), arrived at the same requirements (13).

### 3. First integrals associated with general dynamical symmetries

We consider, with Hojman, the example of a multidimensional oscillator with Lagrangian

$$L=\tfrac{1}{2}\sum_{i=1}^{n}(\dot{q}^{i^2}-q^{i^2}). \tag{16}$$

We then have $\Lambda^i=-q^i$ and a dynamical symmetry is given by

$$Y=E(q^i\,\partial/\partial q^i+\dot{q}^i\,\partial/\partial\dot{q}^i) \tag{17}$$

where $E=\frac{1}{2}\Sigma_i(\dot{q}^{i^2}+q^{i^2})$ is the energy function. The symmetry $Y$ is actually the product of the constant of the motion $E$ with a symmetry satisfying the conditions (13), but does not itself satisfy these conditions and hence does not give rise to a new function $L'$. Finding associated first integrals for such a symmetry was the main point in Hojman's paper. Though his approach certainly is original and has its own merits, in fact the same question was solved before by Crampin (1983a) along the following lines (see also Sarlet and Cantrijn 1984). Putting

$$\alpha=i_Y\,\mathrm{d}\theta_L, \tag{18}$$

the relations

$$\left.\begin{aligned} &i_{R(X)}\,\mathrm{d}\theta_L=i_X\,\mathrm{d}\alpha \quad (19)\\ &\langle R(X),\mathrm{d}t\rangle=0 \quad (20)\end{aligned}\right\}\quad\text{for all vector fields } X$$

under the assumption that $L$ is regular, uniquely define a type $(1,1)$ tensor field $R$, here considered as a linear map (over the ring of functions) on the set of vector fields. What makes $R$ interesting is the property

$$\mathscr{L}_\Gamma R=0. \tag{21}$$

The coordinate form of this property gives rise to a matrix equation of Lax type and the eigenvalues, as well as the other invariants of the matrix, are therefore first integrals. Crampin *et al* (1983) subsequently succeeded in further characterising a situation in which the system eventually turns out to be completely integrable.

Returning to our example, we have

$$\mathrm{d}\theta_L=\mathrm{d}\dot{q}^i\wedge\mathrm{d}q^i-\mathrm{d}E\wedge\mathrm{d}t \tag{22}$$

and $\alpha$, as defined by (18), explicitly reads

$$\alpha=E(\dot{q}^i\,\mathrm{d}q^i-q^i\,\mathrm{d}\dot{q}^i-2E\,\mathrm{d}t). \tag{23}$$

Now let $X$ be an arbitrary vector field, locally represented as

$$X=\rho\,\partial/\partial t+\alpha^i\,\partial/\partial q^i+\beta^i\,\partial/\partial\dot{q}^i. \tag{24}$$

The right-hand side of equation (19) then becomes

$$\begin{aligned} i_X\,\mathrm{d}\alpha=&(\alpha^iq^i+\beta^i\dot{q}^i)(\dot{q}^j\,\mathrm{d}q^j-q^j\,\mathrm{d}\dot{q}^j-2E\,\mathrm{d}t)-(\alpha^i\dot{q}^i-\beta^iq^i-2E)(q^j\,\mathrm{d}q^j+\dot{q}^j\,\mathrm{d}\dot{q}^j)\\ &+2E(\beta^i+q^i\rho)\,\mathrm{d}q^i-2E(\alpha^i-\dot{q}^i\rho)\,\mathrm{d}\dot{q}^i-(\beta^i\dot{q}^i+\alpha^iq^i)\,\mathrm{d}t.\end{aligned} \tag{25}$$

If, on the other hand, we write $R(X)$ as

$$R(X)=\xi^i\,\partial/\partial q^i+\eta^i\,\partial/\partial\dot{q}^i \tag{26}$$

with zero $\partial/\partial t$ component in view of (20), the left-hand side of (19) takes the form

$$i_{R(X)}\,\mathrm{d}\theta_L = \eta^i\,\mathrm{d}q^i - \xi^i\,\mathrm{d}\dot{q}^i - (\eta^i\dot{q}^i + \xi^i q^i)\,\mathrm{d}t. \tag{27}$$

Identifying the coefficients of $\mathrm{d}q^i$ and $\mathrm{d}\dot{q}^i$ in (25) with those in (27), we obtain an explicit expression for the linear transformation from the vector $(\rho, \alpha^i, \beta^i)$ to the vector $(0, \xi^i, \eta^i)$, from which it is easy to read directly the coordinate expression of the (1, 1) tensor $R$ (identification of the $\mathrm{d}t$ terms in (25) and (27) merely produces a relation which is identically satisfied from the previous ones). We thus find $R$ to be

$$R = (q^i q^j + \dot{q}^i\dot{q}^j + 2E\delta^i_j)\left(\frac{\partial}{\partial q^i}\otimes \mathrm{d}q^j + \frac{\partial}{\partial \dot{q}^i}\otimes \mathrm{d}\dot{q}^j\right) + (q^i\dot{q}^j - \dot{q}^i q^j)\left(\frac{\partial}{\partial q^i}\otimes \mathrm{d}\dot{q}^j - \frac{\partial}{\partial \dot{q}^i}\otimes \mathrm{d}q^j\right)$$
$$-4E\dot{q}^i\frac{\partial}{\partial q^i}\otimes \mathrm{d}t + 4Eq^i\frac{\partial}{\partial \dot{q}^i}\otimes \mathrm{d}t. \tag{28}$$

The coefficients in the first two terms of (28) determine the earlier mentioned $2n\times 2n$ matrix with constant eigenvalues and this matrix is seen to be identical to the one obtained by Hojman (see Hojman (1984), equation (5.15)). Having convinced ourselves through this example that Hojman's results are indeed covered by Crampin's procedure (19), (20), we can, for the general theory, actually obtain formulae which are slightly better than those given by Hojman. The reason is that, while the relevant coefficient matrix in $R$, in principle, results from the product of one $2n\times 2n$ matrix with the inverse of another one (see e.g. Hojman's equation (4.19)), it should be possible to obtain a formula in which only the inverse of the $n\times n$ Hessian matrix (14) is involved. A suitable basis of 1-forms and vector fields for this purpose has been provided by Crampin (1983b) for the autonomous case and by Crampin *et al* (1984) for the time-dependent theory.

Going back to the general situation of system (1) and the associated vector field $\Gamma$ in (2), we thus introduce the dual bases $\{\theta^i, \psi^i, \mathrm{d}t\}$ and $\{H_i, V_i, \Gamma\}$ of 1-forms and vector fields, where

$$\theta^i = \mathrm{d}q^i - \dot{q}^i\,\mathrm{d}t \qquad \psi^i = \mathrm{d}\dot{q}^i - \Lambda^i\,\mathrm{d}t + A^i_j\theta^j \tag{29}$$

$$H_i = \partial/\partial q^i - A^j_i\,\partial/\partial\dot{q}^j \qquad V_i = \partial/\partial\dot{q}^i \tag{30}$$

with

$$A^i_j = -\tfrac{1}{2}\,\partial\Lambda^i/\partial\dot{q}^j. \tag{31}$$

Assuming that (1) is derivable from a Lagrangian $L$, $\mathrm{d}\theta_L$ now takes the particularly simple form

$$\mathrm{d}\theta_L = \alpha_{ij}\psi^i\wedge\theta^j. \tag{32}$$

Let $Y$, as in (9), denote a general dynamical symmetry of (1). In view of (11), the 1-form $\alpha$ of (18) will not contain $\mathrm{d}t$ terms. We write

$$\alpha = i_Y\,\mathrm{d}\theta_L = a_j\theta^j - b_j\psi^j \tag{33}$$

where

$$a_j = \alpha_{ij}(\nu^i + A^i_k\mu^k) \qquad b_i = \alpha_{ij}\mu^j. \tag{34}$$

In computing $\mathrm{d}\alpha$, it is quite easy to show that the coefficients of $\mathrm{d}t\wedge\theta^j$ and $\mathrm{d}t\wedge\psi^j$ are zero if one takes account of the following relations, which constitute part of the

Helmholtz conditions (see Sarlet (1982) or Crampin *et al* (1984)) satisfied by $(\alpha_{ij})$:

$$\Gamma(\alpha_{ij}) = \alpha_{ik}A^k_j + \alpha_{jk}A^k_i \qquad \alpha_{ij} = \alpha_{ji} \qquad \alpha_{ij}\Phi^j_k = \alpha_{kj}\Phi^j_i \tag{35}$$

where

$$\Phi^j_k = B^j_k - A^j_l A^l_k - \Gamma(A^j_k) \qquad B^j_k = -\partial\Lambda^j/dq^k. \tag{36}$$

It follows that

$$\mathrm{d}\alpha = \{H_k(a_j) - b_i H_k(A^i_j)\}\theta^k \wedge \theta^j - V_k(b_i)\psi^k \wedge \psi^i + \{V_k(a_j) + H_j(b_k) - b_i V_k(A^i_j)\}\psi^k \wedge \theta^j. \tag{37}$$

We now proceed as in the example above. Writing a general vector field $X$ in the form $X = \rho\Gamma + \alpha^i H_i + \beta^i V_i$ and denoting $R(X)$ by $R(X) = \xi^i H_i + \eta^i V_i$, we can compute both sides of (19) and identify the coefficients of the basis 1-forms. Setting $(g^{ij}) = (\alpha_{ij})^{-1}$, we find eventually that the $(1, 1)$ tensor field $R$, with respect to the indicated dual bases, is given by

$$\begin{aligned} R = g^{lk}\{&[V_k(a_j) + H_j(b_k) - b_i V_k(A^i_j)]H_l \otimes \theta^j + [V_j(a_k) + H_k(b_j) - b_i V_j(A^i_k)]V_l \otimes \psi^j \\ &+ [H_j(a_k) - b_i H_j(A^i_k) - H_k(a_j) + b_i H_k(A^i_j)]V_l \otimes \theta^j \\ &+ [V_j(b_k) - V_k(b_j)]H_l \otimes \psi^j\}. \end{aligned} \tag{38}$$

The $2n \times 2n$ coefficient matrix, which is determined by this expression, constitutes a more explicit version of Hojman's equation (4.19) for reasons indicated above.

As a final remark we recall that Hojman announced a further constant of motion (Hojman (1984), equation (4.24)), which, although not very clearly stated in the paper, only makes sense when two independent symmetries are known (S Hojman, private communication). In our notation, it reads

$$J = -[\Gamma(\alpha_{ij}\mu^j_1) + \alpha_{kj}\mu^j_1\, \partial\Lambda^k/\partial\dot{q}^i]\mu^i_2 + \alpha_{ij}\mu^j_1 \nu^i_2 \tag{39}$$

where $(\mu^j_i, \nu^j_i)$ are the components of the symmetry $Y_i$ $(i = 1, 2)$. Again, this first integral is not essentially related to the use of acceleration dependent Lagrangians. As a matter of fact, one can easily verify that $J = i_{Y_1} i_{Y_2}\, \mathrm{d}\theta_L$, the constancy of which is obvious and was exploited, for example, in Sarlet (1983).

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